MODULE 2

Functions of Two and More Variables and Special Functions

Engineering Mathematics — MTH1001

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Syllabus Coverage (9 hrs.)

Topic
Functions of two or more variables: limit, continuity and differentiability
Homogeneous functions and Euler's theorem
Higher order partial derivatives and Taylor's series
Maximum and minimum values (functions of two variables)
Beta, Gamma functions and Error functions

Chapter 1: Functions of Two or More Variables — Limit, Continuity and Differentiability

A function of two variables assigns a unique real number $z = f(x, y)$ to every ordered pair (x, y) in some domain $D \subseteq \mathbb{R}^2$. These ideas extend naturally to three or more variables, e.g. $w = f(x, y, z)$.

1.1 Neighbourhood and Domain

A δ-neighbourhood of a point (a, b) is the set of all points (x, y) such that:

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

Unlike a single variable, where x can approach c only from the left or the right, (x, y) can approach (a, b) along infinitely many paths — straight lines, parabolas, or any curve. This is the key difference that makes limits of several variables harder to establish.

1.2 Limit of a Function of Two Variables

$f(x, y)$ is said to tend to the limit L as $(x, y) \to (a, b)$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

Written as:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

> **Note:** For the limit to exist, $f(x, y)$ must approach the same value L along every possible path of approach to (a, b). If two different paths give two different values, the limit does not exist.

Worked Example 1 — Limit does not exist

Show that $\lim_{(x,y)\to(0,0)} xy / (x^2 + y^2)$ does not exist.

Solution: The limit is examined along the family of straight-line paths $y = mx$ through the origin. As $(x, y) \to (0, 0)$ along $y = mx$, we have $x \to 0$, so:

$$\lim \ xy/(x^2+y^2) \ = \ \lim \ (mx^2)/[x^2(1+m^2)] \ = \ m/(1+m^2)$$

This value depends on the slope m of the approach path — different lines give different limiting values (e.g. $m = 0$ gives 0, $m = 1$ gives 1/2). Since the limit is not unique, it does not exist.

Worked Example 2 — Limit exists

Examine $\lim_{(x,y)\to(0,0)} x^2y / (x^2 + y^2)$.

Solution: Using polar coordinates $x = r \cos\theta$, $y = r \sin\theta$, as $(x, y) \to (0, 0)$, $r \to 0$ for every direction θ:

$$x^2y/(x^2+y^2) = r^3\cos^2\theta \sin\theta / r^2 = r \cos^2\theta \sin\theta$$

Since $|\cos^2\theta \sin\theta| \le 1$, the expression is bounded by r, which tends to 0 irrespective of θ. Hence the limit equals 0 and exists (the polar-coordinate substitution is a standard trick to test limits of two-variable functions).

1.3 Continuity of a Function of Two Variables

$f(x, y)$ is continuous at (a, b) if:

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

This requires three things simultaneously: (i) f(a, b) must be defined, (ii) the limit must exist (i.e. be the same along every path), and (iii) the limit must equal the function value. A function is continuous in a region R if it is continuous at every point of R.

1.4 Partial Derivatives and Differentiability

The partial derivative of f(x, y) with respect to x, treating y as constant, is:

$$\partial f/\partial x = f_x = \lim_{h \to 0} [f(x+h, y) - f(x, y)] / h$$

Similarly, the partial derivative with respect to y, treating x as constant, is:

$$\partial f/\partial y = f_y = \lim_{k \to 0} [f(x, y+k) - f(x, y)] / k$$

> **Note:** The mere existence of f_x and f_y at a point does NOT guarantee that f is continuous, let alone differentiable, at that point. Differentiability of a function of two variables is a strictly stronger condition than the mere existence of its partial derivatives.

Differentiability — Formal Definition

f(x, y) is said to be differentiable at (a, b) if the increment $\Delta f = f(a+h, b+k) - f(a, b)$ can be written as:

$$\Delta f = h \cdot f_x(a,b) + k \cdot f_y(a,b) + h \cdot \varepsilon_1 + k \cdot \varepsilon_2$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(h, k) \to (0, 0)$. If f is differentiable at a point, it is automatically continuous there, and both partial derivatives exist. The converse is not true.

1.5 Quick Reference — Chapter 1

Condition	Implication
f differentiable at (a,b)	f continuous at (a,b); f_x, f_y exist
f_x, f_y exist at (a,b)	Does NOT imply continuity or differentiability
f continuous at (a,b)	Does NOT imply f_x, f_y exist
Limit along all paths equal	Limit exists
Limit differs along two paths	Limit does not exist

Chapter 2: Homogeneous Functions and Euler's Theorem

2.1 Homogeneous Function — Definition

A function f(x, y) is said to be a homogeneous function of degree n if, for every value of t:

$$f(tx, ty) = t^n f(x, y)$$

Equivalently, f(x, y) is homogeneous of degree n if it can be written as:

$$f(x, y) = x^n \cdot \varphi(y/x) \quad \text{or} \quad f(x, y) = y^n \cdot \psi(x/y)$$

Examples of Homogeneous Functions

Function f(x, y)	Degree n	Reasoning
$x^3 + y^3 + 3xy^2$	3	Every term has total degree 3
$x^2y^2 / (x + y)$	3	Degree of numerator 4 − degree of denominator 1
$\sqrt{(x^2 + y^2)}$	1	$f(tx,ty) = t\cdot\sqrt{(x^2+y^2)}$
$\sin^{-1}(x/y)$	0	$f(tx,ty) = \sin^{-1}(x/y) = f(x,y)$
$\log(y/x)$ or $\tan^{-1}(y/x)$	0	Depends only on the ratio y/x

Note: Functions such as $\log(y/x)$ and $\tan^{-1}(y/x)$ are homogeneous of degree 0, but Euler's theorem is more conveniently applied to them via u = log f or u = f itself, as shown below.

2.2 Euler's Theorem on Homogeneous Functions

Statement: If f(x, y) is a homogeneous function of degree n and is differentiable, then:

$$x \cdot \partial f/\partial x \; + \; y \cdot \partial f/\partial y \; = \; n \cdot f$$

Proof (Sketch)

Since f is homogeneous of degree n, write $f(x, y) = x^n \varphi(y/x)$. Differentiate partially with respect to x and y, using the chain rule for $\varphi(v)$ where $v = y/x$, and then form $x\cdot fx + y\cdot fy$. The terms involving $\varphi'(v)$ cancel, leaving exactly $n\cdot x^n\varphi(v) = n\cdot f(x, y)$, which proves the result.

Extension to Three Variables

If f(x, y, z) is homogeneous of degree n, Euler's theorem extends to:

$$x\cdot\partial f/\partial x + y\cdot\partial f/\partial y + z\cdot\partial f/\partial z = n\cdot f$$

2.3 Euler's Theorem for Second-Order Derivatives

If f(x, y) is homogeneous of degree n, applying Euler's theorem to the first partial derivatives (each of degree n − 1) gives the extended result:

$$x^2\cdot fxx + 2xy\cdot fxy + y^2\cdot fyy = n(n - 1)\cdot f$$

2.4 Worked Examples

Example 1

If $u = x^3 + y^3 + z^3 - 3xyz$, verify Euler's theorem.

Solution: u is homogeneous of degree n = 3 (every term has total degree 3). Computing the partial derivatives:

$$ux = 3x^2 - 3yz,\ \ uy = 3y^2 - 3xz,\ \ uz = 3z^2 - 3xy$$

Then:

$$x \cdot ux + y \cdot uy + z \cdot uz = 3x^3 + 3y^3 + 3z^3 - 9xyz = 3(x^3+y^3+z^3-3xyz) = 3u$$

which confirms $x \cdot ux + y \cdot uy + z \cdot uz = nu$ with $n = 3$.

Example 2 — Using u = log f (degree-0 case)

If $u = \log[(x^3 + y^3)/(x + y)]$, show that $x \cdot ux + y \cdot uy = 2$.

Solution: Let $f = (x^3+y^3)/(x+y) = e^u$. Here f is homogeneous of degree 2 (since numerator is degree 3 and denominator is degree 1). By Euler's theorem applied to f:

$$x \cdot fx + y \cdot fy = 2f$$

Since $f = e^u$, $fx = e^u \cdot ux$ and $fy = e^u \cdot uy$. Substituting:

$$x \cdot e^u \cdot ux + y \cdot e^u \cdot uy = 2e^u \implies x \cdot ux + y \cdot uy = 2$$

Note: This substitution trick — writing u = log f and applying Euler's theorem to $f = e^u$ — is the standard technique whenever the given expression is log or $\tan^{-1}$ of a homogeneous ratio.

Example 3 — Second-order Euler relation

If $u = x^2y^2 \sin^{-1}(y/x)$, verify $x^2uxx + 2xy \cdot uxy + y^2uyy = n(n-1)u$ for the appropriate n.

Solution: Here $u = x^2y^2 \cdot \sin^{-1}(y/x)$. Since x^2y^2 is homogeneous of degree 4 and $\sin^{-1}(y/x)$ is homogeneous of degree 0, u is homogeneous of degree n = 4. By the second-order Euler relation:

$$x^2uxx + 2xy \cdot uxy + y^2uyy = 4(4 - 1)u = 12u$$

Chapter 3: Higher Order Partial Derivatives and Taylor's Series

3.1 Higher Order Partial Derivatives

Since fx and fy are themselves functions of x and y, they can be differentiated again, giving four second-order partial derivatives:

Notation	Meaning
$f_{xx} = \partial^2 f/\partial x^2$	Differentiate f w.r.t. x, twice
$f_{yy} = \partial^2 f/\partial y^2$	Differentiate f w.r.t. y, twice
$f_{xy} = \partial^2 f/\partial y\partial x$	Differentiate w.r.t. x first, then y
$f_{yx} = \partial^2 f/\partial x\partial y$	Differentiate w.r.t. y first, then x

Equality of Mixed Partial Derivatives (Young's / Schwarz's Theorem)

If fxy and fyx are both continuous in a neighbourhood of (a, b), then they are equal at that point:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Note: This result is what allows mixed partial derivatives to be computed in either order for all standard elementary functions encountered in engineering mathematics.

3.2 Taylor's Series for a Function of Two Variables

If f(x, y) and its partial derivatives up to order n are continuous in a neighbourhood of (a, b), then:

$$f(a+h, b+k) = f(a,b) + [h\cdot\partial/\partial x + k\cdot\partial/\partial y]\, f(a,b) + (1/2!)[h\cdot\partial/\partial x + k\cdot\partial/\partial y]^2\, f(a,b) + (1/3!)[h\cdot\partial/\partial x + k\cdot\partial/\partial y]^3\, f(a,b) + \ldots$$

Here the operator $[h\cdot\partial/\partial x + k\cdot\partial/\partial y]^n$ is expanded symbolically using the binomial theorem and then applied to f, evaluated at (a, b). For instance, the second-order term expands as:

$$(1/2!)[h^2\cdot f_{xx} + 2hk\cdot f_{xy} + k^2\cdot f_{yy}] \text{ evaluated at } (a, b)$$

3.3 Maclaurin's Series for Two Variables (Special Case a = 0, b = 0)

$$f(h, k) = f(0,0) + [h\cdot f_x + k\cdot f_y] + (1/2!)[h^2 f_{xx} + 2hk\cdot f_{xy} + k^2 f_{yy}] + \ldots \quad \text{(all derivatives at (0,0))}$$

3.4 Worked Example

Expand $f(x, y) = e^x \log(1 + y)$ in powers of x and y up to third-degree terms.

Solution: This is a Maclaurin expansion about (0, 0). Compute the required partial derivatives at the origin:

Derivative	Value at (0,0)
f	$e^x \log(1+y) \to 0$
fx	$e^x \log(1+y) \to 0$
fy	$e^x/(1+y) \to 1$
fxx	$e^x \log(1+y) \to 0$
fxy	$e^x/(1+y) \to 1$

Derivative	Value at (0,0)
fyy	$-e^x/(1+y)^2 \rightarrow -1$

Substituting into the Maclaurin series:

$$e^x \log(1+y) = y + xy - y^2/2 + \ldots \quad \textit{(up to third-degree terms)}$$

Chapter 4: Maximum and Minimum Values of Functions of Two Variables

4.1 Stationary (Critical) Points

A point (a, b) is a stationary point of f(x, y) if both first-order partial derivatives vanish there:

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

Note: As with a single variable, the vanishing of fx and fy is a necessary but not sufficient condition for an extreme value — it locates candidates only. A saddle point also satisfies fx = fy = 0 without being a maximum or minimum.

4.2 Second Derivative Test (rt − s² test)

At a stationary point (a, b), define:

$$r = f_{xx}(a,b), \quad s = f_{xy}(a,b), \quad t = f_{yy}(a,b)$$

Condition	Conclusion
$rt - s^2 > 0$ and $r < 0$	f has a maximum value at (a, b)
$rt - s^2 > 0$ and $r > 0$	f has a minimum value at (a, b)
$rt - s^2 < 0$	(a, b) is a saddle point — neither max nor min
$rt - s^2 = 0$	Test is inconclusive; further investigation needed

4.3 Worked Example 1

Find the extreme values of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

Solution: The first partial derivatives are:

$$f_x = 3x^2 - 3 = 0 \implies x = \pm 1, \qquad f_y = 3y^2 - 12 = 0 \implies y = \pm 2$$

This gives four stationary points: (1, 2), (1, −2), (−1, 2), (−1, −2). The second derivatives are:

$$f_{xx} = 6x = r, \quad f_{xy} = 0 = s, \quad f_{yy} = 6y = t$$

Point	r = 6x	t = 6y	$rt - s^2$	Conclusion
(1, 2)	6	12	72 > 0, r > 0	Minimum, f = 2
(1, −2)	6	−12	−72 < 0	Saddle point
(−1, 2)	−6	12	−72 < 0	Saddle point
(−1, −2)	−6	−12	72 > 0, r < 0	Maximum, f = 38

4.4 Worked Example 2

Divide 24 into three parts such that the continued product of the first, the square of the second, and the cube of the third is maximum.

Solution: Let the three parts be x, y, 24 − x − y. Maximize:

$$f(x, y) = x \cdot y^2 \cdot (24 - x - y)^3$$

Setting $f_x = 0$ and $f_y = 0$ and solving simultaneously (after standard algebraic simplification) yields the stationary point $x = 4$, $y = 8$, so the third part is 12. Applying the $rt - s^2$ test at this point confirms $r < 0$ and $rt - s^2 > 0$, so this gives the maximum product, with parts in the ratio 4 : 8 : 12 = 1 : 2 : 3.

4.5 Lagrange's Method of Undetermined Multipliers

To find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$, form the auxiliary function:

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda \cdot g(x, y, z)$$

The extreme values occur where all of the following hold simultaneously:

$$\partial F/\partial x = 0, \quad \partial F/\partial y = 0, \quad \partial F/\partial z = 0, \quad g(x, y, z) = 0$$

λ is called the Lagrange multiplier. Solving these equations simultaneously gives the critical points subject to the constraint, without needing to eliminate a variable explicitly.

Worked Example — Lagrange's Multipliers

Find the maximum value of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 3a$.

Solution: Form $F = xyz + \lambda(x + y + z - 3a)$. Then:

$$F_x = yz + \lambda = 0, \quad F_y = xz + \lambda = 0, \quad F_z = xy + \lambda = 0$$

From the first two equations, $yz = xz \Longrightarrow x = y$ (for $z \neq 0$); similarly $y = z$. Combined with the constraint $x + y + z = 3a$, this gives $x = y = z = a$. Hence the maximum value is:

$$f(a, a, a) = a^3$$

Chapter 5: Gamma, Beta and Error Functions

Many definite integrals that cannot be expressed using elementary functions can be evaluated in closed form using the Gamma, Beta and Error functions. These are indispensable tools in probability, statistics and engineering analysis.

5.1 Gamma Function

The Gamma function, denoted Γ(p), also called Euler's integral of the second kind, is defined by the improper integral:

$$\Gamma(p) = \int_0^\infty e^{-t}\, t^{p-1}\, dt, \quad (p > 0)$$

Key Properties

Property	**Statement**
Reduction formula	$\Gamma(p + 1) = p \cdot \Gamma(p)$
Factorial relation	$\Gamma(n + 1) = n!$ for positive integer n
Γ(1)	$\Gamma(1) = 1$
Γ(1/2)	$\Gamma(1/2) = \sqrt{\pi}$
General form	$\Gamma[(p+1)/q] = q \cdot a^{((p+1)/q)} \int_0^\infty x^p e^{-ax^q} dx$

Note: Since $\Gamma(p+1) = p \cdot \Gamma(p)$, the Gamma function generalises the factorial to non-integer and even complex values of p. This is what allows expressions like $\Gamma(5/2)$ or $\Gamma(-1/2)$ to be evaluated.

5.2 Beta Function

The Beta function, β(p, q), also called Euler's integral of the first kind, is defined by:

$$\beta(p, q) = \int_0^1 x^{p-1} (1 - x)^{q-1}\, dx, \quad (p > 0, q > 0)$$

Key Properties

Property	**Statement**
Symmetry	$\beta(p, q) = \beta(q, p)$
Trigonometric form	$\beta(p, q) = 2\int_0^{\pi/2} \sin^{(2p-1)}\theta \cdot \cos^{(2q-1)}\theta \, d\theta$
Relation to Γ	$\beta(p, q) = \Gamma(p)\cdot\Gamma(q) / \Gamma(p + q)$
Recurrence	$\beta(p, q) = \beta(p+1, q) + \beta(p, q+1)$
Positive integers	$\beta(m, n) = (m-1)!(n-1)! / (m+n-1)!$

Worked Example — Relation between Beta and Gamma

Prove that $\beta(p, q) = \Gamma(p)\Gamma(q) / \Gamma(p+q)$.

Solution: Starting from $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\, dx$, substitute $x = t^2$, $dx = 2t\, dt$ to obtain $\Gamma(p) = 2\int_0^\infty t^{(2p-1)} e^{-t^2} dt$. Forming the product $\Gamma(p)\Gamma(q)$ as a double integral and converting to polar coordinates ($x = r \cos\theta$, $y = r \sin\theta$) separates the integral into an angular part — which is $(1/2)\beta(p, q)$ — and a radial part, which reduces to $(1/2)\Gamma(p+q)$ after the substitution $r^2 = t$. Combining these two results gives $\Gamma(p)\Gamma(q) = \beta(p, q)\cdot\Gamma(p + q)$, which is the required relation.

5.3 Error Function and Complementary Error Function

The Error function, erf(x), arises from the Gaussian (normal) probability integral and is defined by:

$$erf(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$$

The Complementary Error function is defined as:

$$erfc(x) = 1 - erf(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-t^2} dt$$

Key Properties of erf(x)

Property	Value / Statement
erf(0)	0
erf(∞)	1
Odd function	erf(−x) = −erf(x)
Relation to Γ(1/2)	$\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$, consistent with $\Gamma(1/2) = \sqrt{\pi}$
erf(x) + erfc(x)	= 1 for all x

Note: The error function has no elementary closed form and is always expressed either as the defining integral or read from tabulated / series values; it plays the same role for the Gaussian distribution that trigonometric functions play for oscillatory phenomena.

5.4 Quick Reference — Chapter 5

Function	Formula
Gamma	$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$
Gamma (integer)	$\Gamma(n+1) = n!$
Beta	$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$
Beta–Gamma relation	$\beta(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$
Error function	$erf(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2} dt$
Complementary error function	$erfc(x) = 1 - erf(x)$

Quick Reference — All Key Formulae (Module 2)

Limits, Continuity, Differentiability

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ requires the same limit along every path of approach}$$

$$f \text{ continuous at } (a,b) \iff \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$$\text{Differentiable} \Longrightarrow \text{Continuous} \Longrightarrow / \Longleftarrow f_x, f_y \text{ exist}$$

Euler's Theorem

$$f(tx,ty) = t^n f(x,y) \Longrightarrow x \cdot f_x + y \cdot f_y = n \cdot f$$

$$x^2 f_{xx} + 2xy \cdot f_{xy} + y^2 f_{yy} = n(n-1) \cdot f$$

Taylor's / Maclaurin's Series (Two Variables)

$$f(a+h,b+k) = f(a,b) + [h \cdot \partial x + k \cdot \partial y]f + (1/2!)[h \cdot \partial x + k \cdot \partial y]^2 f + \ldots$$

Maxima and Minima

$$\text{Stationary point: } f_x = 0,\ f_y = 0$$

$$r=f_{xx},\ s=f_{xy},\ t=f_{yy};\ rt-s^2>0 \ \&\ r<0 \Longrightarrow \text{Max};\ rt-s^2>0 \ \&\ r>0 \Longrightarrow \text{Min};\ rt-s^2<0 \Longrightarrow \text{Saddle}$$

$$\text{Lagrange: } F = f + \lambda g,\ F_x=F_y=F_z=0,\ g=0$$

Beta, Gamma, Error Functions

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt, \quad \Gamma(n+1)=n!, \quad \Gamma(1/2)=\sqrt{\pi}$$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx = \Gamma(p)\Gamma(q)/\Gamma(p+q)$$

$$\text{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2} dt, \quad \text{erfc}(x) = 1 - \text{erf}(x)$$

— End of Module 2 Notes —